

Singular Limits of Anisotropic Reaction-Diffusion Equations with Degenerating Diffusion

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Abstract

We study the long-time behavior of solutions to a Cauchy problem for an anisotropic reaction-diffusion equation of the form

$$u_t(x, t) - \operatorname{Tr}(D_h^2 u(x, t)) + f(u(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty),$$

where f is a cubic nonlinearity with two stable equilibria m_- and m_+ and one unstable equilibrium m_o , satisfying $m_- < m_o < m_+$. The diffusion operator is defined via $m \leq n$ vector fields $\sigma^{(1)}, \dots, \sigma^{(m)}$, generators of a step-two Carnot group, which induces a degenerate parabolic structure. Specifically, we define the *horizontal Hessian matrix* $\tilde{D}_h^2 u(x)$ and its *symmetrized* version $D_h^2 u(x)$ as

$$\tilde{D}_h^2 u(x) := D_h(D_h u(x)) = \left(\sigma^{(i)}(x) D(\sigma^{(j)}(x) Du(x)) \right)_{i,j=1,\dots,m},$$

$$D_h^2 u(x) = \frac{1}{2} \left(\tilde{D}_h^2 u(x) + \tilde{D}_h^2 u(x)^t \right).$$

Setting $\alpha = 2m_o - m_- - m_+$, we analyze the singular limits as $\varepsilon \rightarrow 0^+$ of two approximating problems.

Case 1 ($\alpha \neq 0$):

$$u_t^\varepsilon(x, t) - \varepsilon \operatorname{Tr}(D_{h,\varepsilon}^2 u^\varepsilon(x, t)) + \frac{f(u^\varepsilon(x, t))}{\varepsilon} = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty).$$

Case 2 ($\alpha = 0$):

$$u_t^\varepsilon(x, t) - \operatorname{Tr}(D_{h,\varepsilon}^2 u^\varepsilon(x, t)) + \frac{f(u^\varepsilon(x, t))}{\varepsilon^2} = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty).$$

Here, $D_{h,\varepsilon}^2$ denotes the horizontal Hessian associated with a Riemannian approximation, obtained by augmenting the original m vector fields with n additional fields $\varepsilon^k e^i$, $i = 1, \dots, n$, where $\{e^i\}$ is the canonical basis of \mathbb{R}^n and k is chosen appropriately. This renders the approximating equation uniformly parabolic, with degeneracy recovered in the limit $\varepsilon \rightarrow 0^+$.

Starting from initial data assuming values close to both stable equilibria, we prove that the family of solutions u^ε converges globally in time to the stable equilibria away from a propagating interface (front). The behavior of the front depends on the two cases:

- **Case 1** ($\alpha \neq 0$): the front propagates with a constant subelliptic velocity c , determined by the travelling wave solution of the parabolic equation. The motion is governed by the first-order Hamilton-Jacobi equation

$$w_t(x, t) + c|D_h w(x, t)| = 0, \quad (x, t) \in \mathbb{R}^n \times (0, +\infty),$$

where the front is identified as a level set of w .

- **Case 2** ($\alpha = 0$): the front evolves according to the *subelliptic mean curvature flow* defined by the original m vector fields.

In both cases, convergence is established globally in time via the weak geometric flow framework based on the theory of viscosity solutions for the level-set equation. Uniqueness of the level-set solution plays a crucial role: in the literature, it is established when $\alpha \neq 0$, while it remains an open problem in general for the subelliptic mean curvature flow.